

Test Exam for T6

Thermodynamics and Statistical Physics

2018-2019

Read these instructions carefully before making the exam!

- Write your name and student number on *every* sheet.
- *Make sure to write readable for other people than yourself. Points will NOT be given for answers in illegible writing.*
- *Language; your answers have to be in English.*
- Use a *separate* sheet for each problem.
- Use of a (graphing) calculator is allowed.
- This exam consists of 4 problems.
- The weight of the problems is Problem 1 (P1=25 pts); Problem 2 (P2=20 pts); Problem 3 (P3=20 pts) and Problem 4 (P=25 pts). Weights of the various subproblems are indicated at the beginning of each problem.
- The grade of the exam is calculated as $(P1+P2+P3 +P4+10)/10$.
- For all problems you have to write down your arguments and the intermediate steps in your calculation, *else the answer will be considered as incomplete and points will be deducted.*

PROBLEM 1

Score: $a+b+c+d+e=5+5+5+5+5=25$

The Dieterici equation is an equation of state that is sometimes used to describe a real gas:

$$P(V - b) = RTe^{-\frac{a}{RTV}}$$

in which P, V, T are the pressure, the molar volume and the temperature of the gas, respectively. The constant a controls the attractive molecular interactions and the constant b corrects for the volume of the gas molecules.

- a) Show that for a gas described by the Dieterici equation (a Dieterici gas), the critical temperature, pressure and volume are given by:

$$(T_c, P_c, V_c) = \left(\frac{a}{4Rb}, \frac{a}{4e^2b^2}, 2b\right)$$

- b) Describe what happens if a Dieterici gas is compressed at a temperature that is larger than the critical temperature.

- c) Show that the second and third virial coefficient of a Dieterici gas are given by:

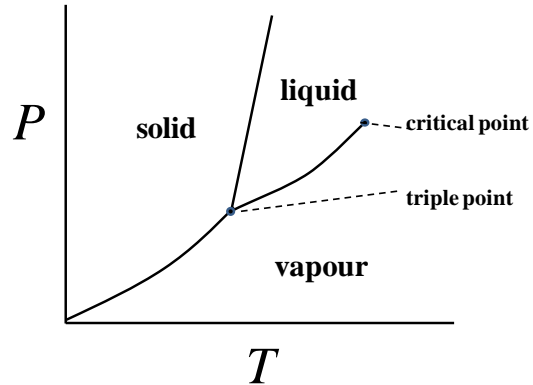
$$B(T) = b - \frac{a}{RT} \quad \text{and} \quad C(T) = b^2 - \frac{ab}{RT} + \frac{a^2}{2(RT)^2}$$

- d) Calculate the Boyle temperature T_b of the Dieterici gas. Why is the name Boyle temperature appropriate?
- e) Use reduced coordinates to rewrite the Dieterici equation. When is it useful to use reduced coordinates?

PROBLEM 2

Score: $a+b+c+d+e=4+5+3+3+5=20$

The figure on the right gives a typical pressure (P) – temperature (T) phase diagram of a one-component system with only one triple point.



The Clausius-Clapeyron equation is:

$$\frac{dP}{dT} = \frac{\Delta S}{\Delta V}$$

- Describe in words when this equation can be applied and explain all the symbols in the equation.
- Derive the Clausius-Clapeyron equation starting from the premises that the Gibbs free energy per molecule (g) in phase 1 and phase 2 is equal in the situation when equilibrium between these phases exists. Thus, $g_1(T, P) = g_2(T, P)$.
- Describe in words what happens at the critical point.
- Use the Clausius-Clapeyron equation to explain that the substance in the phase diagram expands upon melting.

Consider the vapour curve in the phase diagram. Assume that 1) the latent heat L (in Joule per mole) for vaporization is independent of temperature, 2) the vapour obeys the ideal gas law and 3) the volume of the liquid is small compared to the volume of the vapour.

- Show that under these assumptions the vapour pressure is given by:

$$P_{vap} = C e^{-\frac{L}{RT}}$$

where, C is a constant.

PROBLEM 3

Score: $a+b+c+d+e=4+4+4+5+3=20$

Consider a (three dimensional) ideal gas of ultra-relativistic fermions with spin $\frac{1}{2}$ that is confined to a volume V and in equilibrium with a heat bath at temperature T and a particle reservoir with chemical potential μ . For an ultra-relativistic fermion the contribution of its rest mass to the total fermion energy E is negligible with respect to the contribution due to its momentum.

$$E^2 = p^2c^2 + m^2c^4 \approx p^2c^2 \Rightarrow E \approx pc$$

a) Show that for this gas the total number of particles is,

$$N = \frac{8\pi V}{h^3 c^3} \int_0^{\infty} \frac{E^2 dE}{e^{\beta(E-\mu)} + 1}$$

HINT 1: The density of states for a *spinless* particle confined to an enclosure with volume V is (expressed as a function of the particle's momentum p):

$$g(p)dp = \frac{V}{h^3} 4\pi p^2 dp$$

b) Show that at zero absolute temperature ($T = 0$), the maximum energy E_F of an fermion in this gas is,

$$E_F = hc \left(\frac{3N}{8\pi V} \right)^{\frac{1}{3}}$$

c) Show that the total energy U of the ultra-relativistic fermion gas at $T = 0$ is:

$$U = \frac{3}{4} N E_F$$

d) Calculate the pressure of the ultra-relativistic fermion gas at $T = 0$.

e) Explain how fermions at zero absolute temperature can have such high velocities that they are relativistic.

PROBLEM 4 Score: $a+b+c+d+e+f+g = 4+3+4+4+4+3+3=25$

A gas of *non-interacting* bosons in an enclosure with volume V is in contact with both a heat bath at temperature T and a particle reservoir characterized by the chemical potential μ . A state of the gas is described by the set of occupation numbers $n_1, n_2, \dots, n_i, \dots$ of the single-boson states with energies $\varepsilon_1 \leq \varepsilon_2 \leq \dots \leq \varepsilon_i \leq \dots$, respectively.

The grand partition function \mathcal{Z} for this gas of bosons is defined as:

$$\mathcal{Z} = \sum_{n_1, n_2, \dots} e^{\beta[\mu(n_1+n_2+\dots)-(n_1\varepsilon_1+n_2\varepsilon_2+\dots)]}$$

And the probability of finding the gas in the state $n_1, n_2, \dots, n_i, \dots$ is given by:

$$P(n_1, n_2, \dots, n_i, \dots) = \frac{e^{\beta[\mu(n_1+n_2+\dots)-(n_1\varepsilon_1+n_2\varepsilon_2+\dots)]}}{\mathcal{Z}}$$

a) Show that this grand partition function and probability factorize as:

$$\mathcal{Z} = \prod_{i=1}^{\infty} \mathcal{Z}_i \quad \text{with} \quad \mathcal{Z}_i = \sum_{n_i} e^{\beta(\mu-\varepsilon_i)n_i}$$

and

$$P(n_1, n_2, \dots, n_i, \dots) = \prod_{i=1}^{\infty} P_i(n_i) \quad \text{with} \quad P_i(n_i) = \frac{e^{\beta(\mu-\varepsilon_i)n_i}}{\mathcal{Z}_i}$$

b) Give the interpretation of the function $P_i(n_i)$.

c) Show that for bosons we have:

$$\mathcal{Z}_i = \frac{1}{1 - e^{\beta(\mu-\varepsilon_i)}}$$

d) Prove that the mean occupation number $\langle n_i \rangle$ of the i -th single-boson state can be calculated from:

$$\langle n_i \rangle = \frac{1}{\beta} \left(\frac{\partial \ln \mathcal{Z}_i}{\partial \mu} \right)_{T,V}$$

and use this expression to calculate $\langle n_i \rangle$.

e) Show that the total number of bosons N of the gas is given by,

Problem continues on next page

$$N = \frac{1}{e^{-\beta\mu} - 1} + \left[\frac{2\pi V}{h^3} (2m)^{\frac{3}{2}} \right] \int_0^{\infty} \frac{\sqrt{\varepsilon} d\varepsilon}{e^{\beta(\varepsilon-\mu)} - 1}$$

Discuss the origin and the meaning of the first term $N_1 = \frac{1}{e^{-\beta\mu} - 1}$ in the expression above.

HINT: The density of states for a boson confined to an enclosure with volume V is (expressed as a function of the particle's momentum p):

$$g(p)dp = \frac{V}{h^3} 4\pi p^2 dp$$

- f) The critical temperature T_c for the boson gas occurs when $\mu = 0$. What is the physical interpretation of the critical temperature?
- g) Show that the ratio N_1/N can be written as:

$$\frac{N_1}{N} = 1 - \left(\frac{T}{T_c} \right)^{\frac{3}{2}}$$

Solutions

PROBLEM 1

a)

The critical point is found when the isotherm of the gas has an inflection point, thus as,

$$\left(\frac{\partial P}{\partial V}\right)_T = \left(\frac{\partial^2 P}{\partial V^2}\right)_T = 0$$

This gives (together with the Dieterici equation of state) three equations with three unknowns namely:

$$P = \frac{RT}{(V-b)} e^{-\frac{a}{RTV}} \quad \text{eq(1)}$$

$$\begin{aligned} \left(\frac{\partial P}{\partial V}\right)_T &= \frac{-RT}{(V-b)^2} e^{-\frac{a}{RTV}} + \frac{RT}{(V-b)} \left(\frac{a}{RTV^2}\right) e^{-\frac{a}{RTV}} \\ &= \frac{RT}{(V-b)} e^{-\frac{a}{RTV}} \left(\frac{a}{RTV^2} - \frac{1}{(V-b)}\right) = P \left(\frac{a}{RTV^2} - \frac{1}{(V-b)}\right) \end{aligned}$$

Thus $\left(\frac{\partial P}{\partial V}\right)_T = 0$ leads to

$$\left(\frac{a}{RTV^2} - \frac{1}{(V-b)}\right) = 0 \Rightarrow \frac{-2a}{RTV^3} = -\frac{2}{V(V-b)} \quad \text{eq(2)}$$

$$\begin{aligned} \left(\frac{\partial^2 P}{\partial V^2}\right)_T &= \left(\frac{\partial P}{\partial V}\right)_T \left(\frac{a}{RTV^2} - \frac{1}{(V-b)}\right) + P \left(\frac{-2a}{RTV^3} + \frac{1}{(V-b)^2}\right) \\ &= P \left(\frac{-2a}{RTV^3} + \frac{1}{(V-b)^2}\right) \end{aligned}$$

And $\left(\frac{\partial^2 P}{\partial V^2}\right)_T = 0$ leads to

$$\left(\frac{-2a}{RTV^3} + \frac{1}{(V-b)^2}\right) = 0 \quad \text{eq(3)}$$

Substituting equation 2 in 3 leads to

$$\left(-\frac{2}{V(V-b)} + \frac{1}{(V-b)^2}\right) = 0 \Rightarrow V_c = 2b$$

Substituting this in equation 2 gives,

$$\frac{a}{RT(2b)^2} - \frac{1}{(2b - b)} = 0 \Rightarrow T_c = \frac{a}{4bR}$$

and $T_c = \frac{a}{4bR}$ and $V_c = 2b$ in equation 1 gives,

$$P_c = \frac{a}{4b^2e^2}$$

Consequently,

$$(T_c, P_c, V_c) = \left(\frac{a}{4Rb}, \frac{a}{4e^2b^2}, 2b\right)$$

b)

The pressure will increase, the density will increase, but the gas will not become a liquid.

c)

Rewrite the Dieterici equation as:

$$\frac{PV}{RT} = \frac{1}{\left(1 - \frac{b}{V}\right)} e^{-\frac{a}{RTV}}$$

and expand both factors on the right hand side in powers of $\frac{1}{V}$:

$$\frac{PV}{RT} = \left(1 + \frac{b}{V} + \left(\frac{b}{V}\right)^2 + \dots\right) \left(1 - \frac{a}{RTV} + \frac{1}{2}\left(\frac{a}{RTV}\right)^2 - \dots\right) \Rightarrow$$

$$\frac{PV}{RT} = 1 + \left(b - \frac{a}{RT}\right)\frac{1}{V} + \left(\frac{a^2}{2(RT)^2} - \frac{ab}{RT} + b^2\right)\frac{1}{V^2} + \dots$$

Thus,

$$B(T) = b - \frac{a}{RT}$$

and

$$C(T) = b^2 - \frac{ab}{RT} + \frac{a^2}{2(RT)^2}$$

d) The temperature at which the second virial coefficient is zero is called the Boyle temperature.

$$B(T) = 0 \Rightarrow b - \frac{a}{RT} = 0 \Rightarrow T_b = \frac{a}{bR}$$

At this temperature Boyle's law ($PV = \text{constant}$) approximately holds for a real gas.

e)

Use the reduced coordinates $\tilde{P} = \frac{P}{P_c}$, $\tilde{T} = \frac{T}{T_c}$, $\tilde{V} = \frac{V}{V_c}$ in the Dieterici equation to find,

$$\frac{P}{P_c} P_c = \frac{R \frac{T}{T_c} T_c}{\left(\frac{V}{V_c} V_c - b\right)} e^{-\frac{a}{R \frac{T}{T_c} V_c T_c V_c}} \Rightarrow \tilde{P} P_c = \frac{R \tilde{T} T_c}{(\tilde{V} V_c - b)} e^{-\frac{a}{R \tilde{T} \tilde{V} T_c V_c}} \Rightarrow$$

$$\tilde{P} \frac{a}{4e^2 b^2} = \frac{R \tilde{T} \frac{a}{4Rb}}{(\tilde{V} 2b - b)} e^{-\frac{a \frac{4Rb}{a} \frac{1}{2b}}{R \tilde{T} \tilde{V}}} = \frac{a}{4b^2} \frac{\tilde{T}}{(2\tilde{V} - 1)} e^{-\frac{2}{\tilde{T}\tilde{V}}} \Rightarrow$$

$$\tilde{P} = \frac{\tilde{T}}{(2\tilde{V} - 1)} e^2 e^{-\frac{2}{\tilde{T}\tilde{V}}} \Rightarrow \tilde{P} = \frac{\tilde{T} e^{2\left(1 - \frac{1}{\tilde{T}\tilde{V}}\right)}}{(2\tilde{V} - 1)}$$

When using reduced coordinates the behaviour of different gases in these scaled coordinates is often vary similar and thus the unknown behaviour of a certain gas can be predicted from that of a gas of which the behaviour is known. This is called the law of corresponding states.

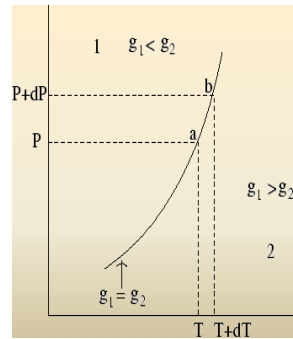
PROBLEM 2

a)

This equation can be applied in the situation of a phase transition between, say, phase 1 and phase 2 of a one-component system. ΔS and ΔV are the change in entropy and volume when a certain amount (1 molecule, 1 gram, 1 mole etc.) of the substance is transformed from phase 1 to phase 2. The equation then relates these changes to the slope of the equilibrium curve in the PT -diagram.

b)

Take two points a and b at the equilibrium curve separated by dT and dP (see figure). Then at point a : $g_1^a = g_2^a$ and at point b : $g_1^b = g_2^b$ which leads to $dg_1 = dg_2$ when moving from point a to point b .



Gibbs free energy is given by,

$$G = E + PV - TS \Rightarrow dG = dE + PdV + VdP - TdS - SdT$$

And using the fundamental thermodynamic relation for a system with variable particle number:

$$dE = TdS - PdV + \mu dN$$

We find,

$$dG = -SdT + VdP + \mu dN$$

Which holds for each phase and using ($\mu = g$) this results in ($i = 1,2$):

$$dG_i = -S_i dT + V_i dP + g_i dN$$

We also have $G_i = N_i g_i \Rightarrow dG_i = N_i dg_i + g_i dN_i$

Combining both expressions for dG_i we derive: $N_i dg_i = -S_i dT + V_i dP$ or $dg_i = -s_i dT + v_i dP$ where s_i and v_i are the entropy and volume per molecule.

Thus,

$$dg_1 = -s_1 dT + v_1 dP = dg_2 = -s_2 dT + v_2 dP \Rightarrow \frac{dP}{dT} = \frac{\Delta s}{\Delta v}$$

which is the Clausius-Clapeyron equation with ΔS and ΔV expressed on a per molecule basis.

c)

If we proceed along the vapour curve to higher temperatures the latent heat of vaporization decreases and becomes zero at the critical point, also the volume change of the phase transition becomes zero at this point. At temperatures and pressure higher than the temperature and pressure of the critical point there is no difference between the liquid and the gas phase.

d)

The Clausius-Clapeyron equation gives the slope of the melting curve. From the figure we see that this slope is positive, In the transformation from solid to liquid the entropy increases ($\Delta S = S_{liquid} - S_{solid} > 0$). For a positive slope we should have $\Delta V = V_{liquid} - V_{solid} > 0$, and thus the volume of the substance increases upon melting.

e)

The entropy change per mole is $\Delta S = \frac{L}{T}$.

The volume change is $\Delta V = V_{vapour} - V_{liquid} \approx V_{vapour}$.

The ideal gas law for one mole of vapour is : $PV_{vapour} = RT$.

Then from the Clausius-Clapeyron equation we find,

$$\frac{dP}{dT} = \frac{\Delta S}{\Delta V} = \frac{L}{T} \frac{1}{V_{vapour}} = \frac{LP}{RT^2} \Rightarrow$$

$$\frac{dP}{P} = \frac{LdT}{RT^2} \Rightarrow d \ln P = d \frac{-L}{RT} \Rightarrow \ln P = \frac{-L}{RT} + C \Rightarrow P = C e^{-\frac{L}{RT}}$$

PROBLEM 3

a)

Use the hint to find the density of states as a function of energy. Remember to multiply with a factor of 2 because of the two spins states of the spin $\frac{1}{2}$ fermion.

$$g(p)dp = \frac{2V}{h^3} 4\pi p^2 dp \xrightarrow[p=\frac{E}{c}]{E} g(E)dE = \frac{2V}{h^3} 4\pi \left(\frac{E}{c}\right)^2 d\left(\frac{E}{c}\right) = \frac{8\pi V}{(hc)^3} E^2 dE$$

Total number of fermions is given by,

$$N = \int_0^{\infty} n(E) g(E) dE$$

with,

$$n(E) = \frac{1}{e^{\beta(E-\mu)} + 1}$$

the mean number of fermions with energy E (Fermi-Dirac distribution)

Thus,

$$N = \frac{8\pi V}{h^3 c^3} \int_0^{\infty} \frac{E^2 dE}{e^{\beta(E-\mu)} + 1}$$

b)

At absolute zero the fermion gas is completely degenerate and all energy levels up to a maximum level E_F (Fermi level) are filled with 1 fermion each and all the other energy levels are empty. Thus, $n(E) = 1$ if $E < E_F$ and $n(E) = 0$ if $E > E_F$. Consequently, the total number of electrons N is given by,

$$N = \frac{8\pi V}{h^3 c^3} \int_0^{E_F} E^2 dE = \frac{8\pi V}{3h^3 c^3} E_F^3 \Rightarrow E_F = hc \left(\frac{3}{8\pi} \frac{N}{V} \right)^{\frac{1}{3}}$$

c)

$$U = \int_0^{\infty} E n(E) g(E) dE = \int_0^{E_F} E g(E) dE \Rightarrow$$

$$U = \frac{8\pi V}{h^3 c^3} \int_0^{E_F} E^3 dE = \frac{8\pi V}{4h^3 c^3} E_F^4 = \frac{3}{4} N E_F$$

d)

$$dU = TdS - PdV + \mu dN \xrightarrow[\substack{T \rightarrow 0 \\ dN=0}]{} P = -\frac{dU}{dV}$$

$$P = -\frac{d}{dV} \left(\frac{3}{4} N E_F \right) = -\frac{d}{dV} \left(\frac{3}{4} N h c \left(\frac{3 N}{8\pi V} \right)^{\frac{1}{3}} \right) = \frac{1}{4} h c \left(\frac{3}{8\pi} \right)^{\frac{1}{3}} \left(\frac{N}{V} \right)^{\frac{4}{3}}$$

e)

Because of the Pauli exclusion principle electrons (fermions) cannot all be in the ground state but occupy different energy levels up to the Fermi energy E_F . When the system is compressed to high density the ratio N/V increases and thus the Fermi energy E_F increases. This implies that also the momenta of the electrons increase possibly up to (ultra) relativistic values. This feature originates from the uncertainty relation: $\Delta p \Delta x \geq \hbar$. When the electron gas is compressed to high density the distance Δx between the electrons decreases, and the momenta Δp of the electrons increase.

PROBLEM 4

a)

$$\mathcal{Z} = \sum_{n_1, n_2, \dots} e^{\beta[\mu(n_1+n_2+\dots)-(n_1\varepsilon_1+n_2\varepsilon_2+\dots)]} = \sum_{n_1, n_2, \dots} e^{\beta(\mu-\varepsilon_1)n_1+\beta(\mu-\varepsilon_2)n_2+\dots} \Rightarrow$$

$$\mathcal{Z} = \sum_{n_1, n_2, \dots} e^{\beta(\mu-\varepsilon_1)n_1} e^{\beta(\mu-\varepsilon_2)n_2} \times \dots = \sum_{n_1} e^{\beta(\mu-\varepsilon_1)n_1} \sum_{n_2} e^{\beta(\mu-\varepsilon_2)n_2} \times \dots \Rightarrow$$

The equality above holds because for each n_i the factor $e^{\beta(\mu-\varepsilon_i)n_i}$ is a constant for all the sums over n_j , with $j \neq i$.

$$\mathcal{Z} = \prod_{i=1}^{\infty} \sum_{n_i} e^{\beta(\mu-\varepsilon_i)n_i} = \prod_{i=1}^{\infty} \mathcal{Z}_i$$

Start with

$$P(n_1, n_2, \dots, n_i, \dots) = \frac{e^{\beta[\mu(n_1+n_2+\dots)-(n_1\varepsilon_1+n_2\varepsilon_2+\dots)]}}{\mathcal{Z}} \Rightarrow$$

$$P(n_1, n_2, \dots, n_i, \dots) = \frac{e^{\beta(\mu-\varepsilon_1)n_1+\beta(\mu-\varepsilon_2)n_2+\dots}}{\mathcal{Z}} \Rightarrow$$

$$P(n_1, n_2, \dots, n_i, \dots) = \frac{e^{\beta(\mu-\varepsilon_1)n_1} e^{\beta(\mu-\varepsilon_2)n_2} \times \dots}{\mathcal{Z}} \Rightarrow$$

$$P(n_1, n_2, \dots, n_i, \dots) = \frac{\prod_{i=1}^{\infty} e^{\beta(\mu-\varepsilon_i)n_i}}{\prod_{i=1}^{\infty} \mathcal{Z}_i} \Rightarrow$$

$$P(n_1, n_2, \dots, n_i, \dots) = \prod_{i=1}^{\infty} P_i(n_i) \text{ with } P_i(n_i) = \frac{e^{\beta(\mu-\varepsilon_i)n_i}}{\mathcal{Z}_i}$$

This means that the probability to find n_i bosons in the i -th single energy state is independent of the occupancies of all other single energy states.

b)

The function $P_i(n_i)$ is the probability of finding n_i bosons in the i -th single boson state.

c)

There can be any number of bosons in each single boson state ($n_i = 0, 1, 2, 3, \dots$) thus,

$$Z_i = \sum_{n_i=0}^{\infty} e^{\beta(\mu-\varepsilon_i)n_i} = \sum_{n_i=0}^{\infty} (e^{\beta(\mu-\varepsilon_i)})^{n_i} = \frac{1}{1 - e^{\beta(\mu-\varepsilon_i)}}$$

d)

The mean occupation number is defined as:

$$\langle n_i \rangle = \sum_{n_i} n_i P_i(n_i)$$

Performing the differentiation:

$$\frac{1}{\beta} \left(\frac{\partial \ln Z_i}{\partial \mu} \right)_{T,V} = \frac{1}{\beta} \frac{\partial}{\partial \mu} \ln \left(\sum_{n_i} e^{\beta(\mu-\varepsilon_i)n_i} \right) = \frac{\sum_{n_i} n_i e^{\beta(\mu-\varepsilon_i)n_i}}{\sum_{n_i} e^{\beta(\mu-\varepsilon_i)n_i}} = \sum_{n_i} n_i \frac{e^{\beta(\mu-\varepsilon_i)n_i}}{Z_i} \Rightarrow$$

$$\frac{1}{\beta} \left(\frac{\partial \ln Z_i}{\partial \mu} \right)_{T,V} = \sum_{n_i} n_i \frac{e^{\beta(\mu-\varepsilon_i)n_i}}{Z_i} = \sum_{n_i} n_i P_i(n_i) = \langle n_i \rangle$$

We now can calculate $\langle n_i \rangle$ as:

$$\langle n_i \rangle = \frac{1}{\beta} \left(\frac{\partial \ln Z_i}{\partial \mu} \right)_{T,V} = \frac{1}{\beta} \frac{\partial}{\partial \mu} \ln \left(\frac{1}{1 - e^{\beta(\mu-\varepsilon_i)}} \right) = \frac{e^{\beta(\mu-\varepsilon_i)}}{1 - e^{\beta(\mu-\varepsilon_i)}} = \frac{1}{e^{\beta(\varepsilon_i-\mu)} - 1}$$

e)

$$N = \int_0^{\infty} g(\varepsilon) n(\varepsilon) d\varepsilon$$

In this $g(\varepsilon)d\varepsilon$ is the density of states and $n(\varepsilon)$ is the mean occupation number.

The density of states follows from the hint and converting momentum to energy (using $= \frac{p^2}{2m}$) as the variable,

Substitute

$$p^2 = 2m\varepsilon \text{ and } 2pdp = 2(2m\varepsilon)^{\frac{1}{2}}dp = 2md\varepsilon \Rightarrow dp = \frac{(2m)^{\frac{1}{2}}}{2\sqrt{\varepsilon}} d\varepsilon$$

in

$$g(p)dp = \frac{V}{h^3} 4\pi p^2 dp$$

to find,

$$g(\varepsilon)d\varepsilon = \frac{V}{h^3} 4\pi 2m\varepsilon \frac{(2m)^{\frac{1}{2}}}{2\sqrt{\varepsilon}} d\varepsilon = \frac{2\pi V}{h^3} (2m)^{\frac{3}{2}} \sqrt{\varepsilon} d\varepsilon$$

The mean occupation number is (from d):

$$n(\varepsilon) = \frac{1}{e^{\beta(\varepsilon-\mu)} - 1}$$

$$N = \int_0^{\infty} \frac{2\pi V}{h^3} (2m)^{\frac{3}{2}} \sqrt{\varepsilon} \frac{d\varepsilon}{e^{\beta(\varepsilon-\mu)} - 1} = \left[\frac{2\pi V}{h^3} (2m)^{\frac{3}{2}} \right] \int_0^{\infty} \frac{\sqrt{\varepsilon}}{e^{\beta(\varepsilon-\mu)} - 1} d\varepsilon$$

However, the ground state $\varepsilon = 0$ has zero weight in this integral because of the $\sqrt{\varepsilon}$ dependency and is completely neglected. This situation can be mended by considering the ground state separately, the occupation number of the ground state is: $N_1 = \frac{1}{e^{-\beta\mu} - 1}$, thus,

$$N = N_1 + \left[\frac{2\pi V}{h^3} (2m)^{\frac{3}{2}} \right] \int_0^{\infty} \frac{\sqrt{\varepsilon}}{e^{\beta(\varepsilon-\mu)} - 1} d\varepsilon$$

f)

At temperatures above the critical temperature T_c the fraction of particles in the ground state is practically zero; as the temperature decreases below T_c the fraction of particles in the ground state increases. These particles have zero energy and zero momentum.

g)

At the critical temperature we have essentially $N_1 = 0$ and $\mu = 0$ and thus (using $z = \beta_c \varepsilon$, in the integral)

$$N = \left[\frac{2\pi V}{h^3} (2m)^{\frac{3}{2}} \right] \int_0^{\infty} \frac{\sqrt{\varepsilon}}{e^{\beta_c \varepsilon} - 1} d\varepsilon = \left[\frac{2\pi V}{h^3} (2m)^{\frac{3}{2}} \right] \left(\frac{1}{\beta_c} \right)^{\frac{3}{2}} \int_0^{\infty} \frac{\sqrt{z}}{e^z - 1} dz \Rightarrow$$

$$N = \left[\frac{2\pi V}{h^3} (2m)^{\frac{3}{2}} \right] \left(\frac{1}{\beta_c} \right)^{\frac{3}{2}} 2.612 \frac{\sqrt{\pi}}{2} \Rightarrow T_c^{\frac{3}{2}} = \frac{N}{\left[\frac{2\pi V}{h^3} (2m)^{\frac{3}{2}} \right] \left(2.612 \frac{\sqrt{\pi}}{2} \right) k^{\frac{3}{2}}}$$

Below the critical temperature the chemical potential is essentially zero and we have for $T < T_c$;

$$N = N_1 + \left[\frac{2\pi V}{h^3} (2m)^{\frac{3}{2}} \right] \int_0^{\infty} \frac{\sqrt{\varepsilon}}{e^{\beta\varepsilon} - 1} d\varepsilon = N_1 + \left[\frac{2\pi V}{h^3} (2m)^{\frac{3}{2}} \right] \left(\frac{1}{\beta} \right)^{\frac{3}{2}} 2.612 \frac{\sqrt{\pi}}{2} \Rightarrow$$

$$N = N_1 + N \left(\frac{\beta_c}{\beta} \right)^{\frac{3}{2}} \Rightarrow \frac{N_1}{N} = 1 - \left(\frac{T}{T_c} \right)^{\frac{3}{2}}$$